ToML

THEORY OF MEASURMENT-LENGTH

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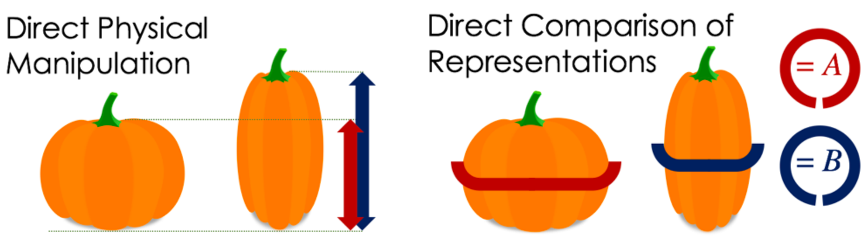
The Theory of Length Measurement (**ToML**) construct describes how children come to constitute a theory of measure to compare magnitudes (extents) of lengths. A theory of measure refers to the web of “big ideas” and procedures involved in developing these comparisons. For example, a measure of 5 inches means that the magnitude of the length is 5 times that of a unit length of 1 inch (a ratio of the magnitudes of the length and the unit length), and this can be verified by 5 iterations (5 translations) of the 1 inch unit. Learning about measurement involves a fusion of practical activity (e.g., how to use tools), the conceptual underpinnings of unit and scale (e.g., units should be identical, the origin of the scale is labeled as zero), and symbolic systems, such as the use of labeled hash marks to designate unit length, and in standard rulers, the use of unlabeled hash marks to designate parts of units. **ToML** is not intended to portray every nuance of learning about the mathematics of length measurement, but instead, to highlight critical conceptual and procedural attainments. Each level is composed of sub-levels that collectively constitute a network of the ideas and procedures that constitute the particular way of knowing and doing expressed in a level. Hence, the sublevels should not be regarded as a ladder, but instead as ideas and procedures that are coordinated to develop the form of thinking described by the overarching level. The levels are arranged to suggest that the ways of thinking described in higher levels encompass the understandings developed in lower levels, but at any one point in time, an individual’s understandings will likely span elements (sub-levels) at multiple levels. Understanding measure of length does not involve replacing one form of knowledge with another, but instead, coordinating the the repertoire of ideas and procedures described at each level.

- Directly Comparing
- Explaining Properties of Units
- Iterating Units and Symbolizing Distance Traveled
- Partitioning By 2
- Partitioning By 3 and 2
- Generalizing Relationships

DIRECTLY COMPARING

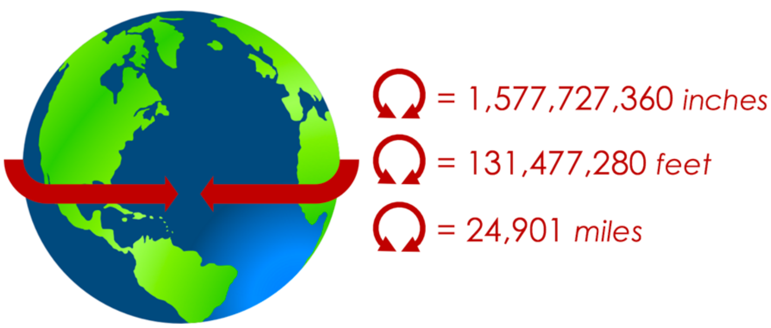
At the entry point of learning, level **ToML-1**, the student identifies potentially measurable attributes of an object, ideally motivated by a sensible question, and compares multiple objects with respect to that attribute. The comparisons at this level are limited to *equal*, *greater than*, and *less than*. To illustrate, children might compare several pumpkins (squashes) with respect to what they call "fatness" (circumference), "bumpiness," "heaviness," and "tallness." The comparisons involving length can be achieved by physically manipulating the objects directly (e.g., by putting two pumpkins next to each other and visually estimating which is tallest), or by directly comparing representations of the attribute, such as paper strips cut to match the length being measured (e.g., the circumference of each pumpkin).



Note that even these simple comparisons include many ideas and procedures important to all measurements. The attribute must be defined in a way that makes its measure feasible and comparable. If groups of children define tallness in a different way, it will be impossible to compare objects on this attribute. For example, some groups might measure from the ground to the top of the stem, while others measure from the ground to the top of the body of the pumpkin. Using tape or paper to measure circumference signals a shift from direct physical to direct representational comparison, an important step toward symbolizing the attribute and its measure.

EXPLAINING PROPERTIES OF UNITS and
HOW THESE PROPERTIES GROUND ACCUMULATION (COUNT)

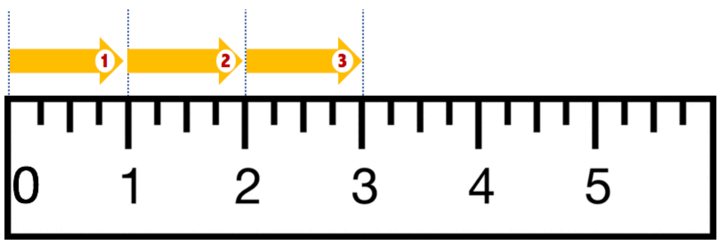
At the second level, **ToML-2**, students consider the nature of a unit. Units enable indirect measurements via accumulation and count. Instead of directly comparing, children who count units produce a measured quantity. Units allow for both additive (how much longer?) and multiplicative comparisons (how many times longer?). But before units can be put to these uses reliably, students must develop understandings of the properties of units that enable these uses. Hence, students at this level explain the roles of identical units and tiling. They measure a length by tiling a length with units that span its extent, explaining that gaps between units will result in unmeasured length (underestimating the true length measure) and that overlaps between units will result in measuring a portion of the length more than once (overestimating the true length measure). Students anticipate an inverse relation between unit length and the measure of a given length: shorter units, compared to longer units, result in a greater measure of the same magnitude (i.e., extent of the length).



Similarly, longer units, compared to shorter units, result in a diminished measure. For example, comparing feet to miles, a fixed distance measured with foot units is always more than the same distance measured with mile units. But at this level, children may not know how many more. Students also begin to develop a sense of the suitability of a particular choice of unit for the goal at hand. For example, yards are better than inches for measuring longer distances. Students also understand the importance of conventional units—for instance, how using the length of their teacher's foot as a shared unit enables comparison of lengths.

ITERATING UNITS and
SYMBOLIZING DISTANCE TRAVELED

ToML-3 is initiated by unit iteration, meaning that counts of units that span a length can be accomplished by translating the unit. (Translation connects measurement to the more general study of isometries—transformations that preserve the lengths and angles of a figure.) As students become familiar with procedures for iterating a unit (e.g., re-using the unit by marking its endpoint to signal the beginning of the next unit), they learn to symbolize the starting point of measure as 0. And they understand conventions, such as labeling the endpoints of units on a ruler to signal distance traveled from the origin (0).



For example, 3 on a standard ruler is marked at the endpoint of the third unit, not at its center. This spacing distinguishes interpretation of a length as distance traveled rather than as merely counting a collection of units. These understandings constitute the beginnings of understanding a measurement scale—a way of specifying relations among units to mark quantities. Students understand that a measure of $10u$ means that the measured length is 10 times as long as the length of 1 unit. Similarly, a measure of $10u$ implies that the length can be subdivided into 10 congruent parts. However, they may not yet routinely understand the reciprocal relation, that 1 unit is $\frac{1}{10}$ times as long as 10 units.

• Directly Comparing
• Explaining Properties of Units
• Iterating Units and Symbolizing Distance Traveled
• Partitioning By 2
• Partitioning By 3 and 2
• Generalizing Relationships

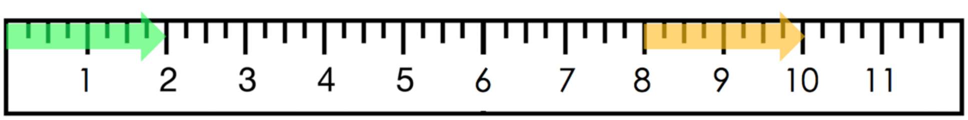
EQUIPARTITIONING by 2 and SYMBOLIZING PARTITIONED UNITS

ZERO-POINT

At **ToML-4**, partitioning units allows the measurement of lengths that may not be measured by a whole number of units. Partitioning arises from everyday actions of breaking or fracturing lengths, such as sticks. But the partitions used for purposes of measure are congruent partitions of units, called equipartitions. At this level, students know how to equipartition unit lengths by 2 and by 4. Students symbolize 2-splits of a unit as $\frac{1}{2}$ *unit*, and they (first) demonstrate and (later) anticipate that $\frac{1}{2}u$ is the only partition of the unit in which 2 iterations of the partitioned unit ($\frac{1}{2}u$) cover the same length as the unit length. Students re-purpose iterative counting with whole numbers of units to iterate by $\frac{1}{2}$, counting $\frac{1}{2}u$, $\frac{2}{2}u$, $\frac{3}{2}u$. (Note: Conceiving of a length as a distance traveled helps students make sense of fractional units greater than one.) Students relate partitions of a unit to the whole unit by coordinating measures in partitioned units with measures in whole units. For instance, they recognize that $\frac{3}{2}u$ is the same distance from the origin of measure as $1\frac{1}{2}u$. (To reason in this way, two levels of unit must be held in mind simultaneously, both the split unit and the unit length.) They understand that 1 unit is 2 times as long as $\frac{1}{2}$ unit, and some students at this level will understand the reciprocal relation as well: $\frac{1}{2}u$ is $\frac{1}{2}$ times as long as 1 *unit*.

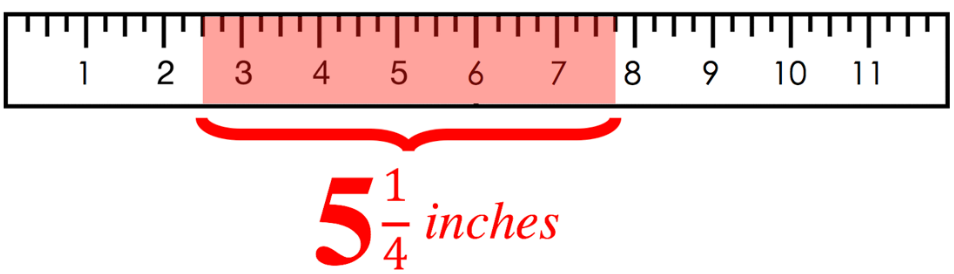
Students first demonstrate and later anticipate that the effects of composing 2-splits of a unit—that is, splitting (partitioning) a unit by 2, and then by 2 again--results in 4 partitions. They explain that $\frac{1}{4}$ is the only partition of the unit for which 4 iterations generate the unit length. Students symbolize 4-splits of a unit as $\frac{1}{4}$ *unit*, and students iterate by $\frac{1}{4}$, counting $\frac{1}{4}u$, $\frac{2}{4}u$, $\frac{3}{4}u$, $\frac{4}{4}u$, $\frac{5}{4}u$. Students relate partitions of a unit to the whole unit by coordinating measures in partitioned units with measures in whole units. For instance, they recognize that $\frac{6}{4}u$ is the same distance from the origin of measure as $1\frac{1}{2}u$. (To do this, two levels of unit must be held in mind simultaneously, both the split unit and the unit length.) Students demonstrate that $\frac{2}{4}$ u and $\frac{1}{2}$ u are the same distance from 0.

Students multiplicatively compare 2- and 4-partitions of a unit to the whole unit. For example, $1u$ is four times as long as $\frac{1}{4}u$, $\frac{3}{2}u$ is 3 times as long as $\frac{1}{2}u$. Some students may recognize reciprocal relations involving 2, such as $\frac{1}{4}u$ is $\frac{1}{4}$ times as long as $1u$ (but usually do not recognize some others, such as $\frac{1}{2}u$ is $\frac{1}{3}$ times as long as $\frac{3}{2}$ unit). As students gain familiarity with symbols and other representations used in measuring tools, such as a ruler, and in marking length, such as line segments, they also come to understand that the origin of measure, the zero point, is arbitrary—so any number will do. At this level, this understanding usually encompasses only whole numbers. For example, on a ruler, a length of 2 *inches* can be measured either as the distance from 0 *in* to 2 *in* or as the distance from 8 *in* to 10 *in*.



PARTITIONING UNITS by 3 and
COMPOSING PARTITIONS OF 2 AND 3

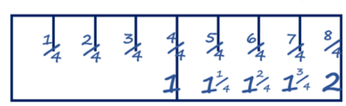
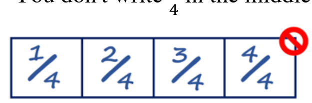
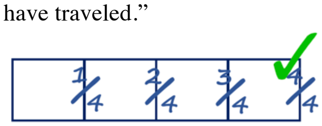
At **ToML-5** students develop 3-splits of units, representing each equipartition as $\frac{1}{3}u$, and iterate by $\frac{1}{3}$, counting $\frac{1}{3}u$, $\frac{2}{3}u$, $\frac{3}{3}u$, $\frac{4}{3}u$. Students understand that $\frac{1}{3}$ *unit* is the only partition of the unit for which 3 iterations generates the unit length. Students compose repeated splits of 2 or of 3 to first demonstrate and then anticipate the number of resulting parts, such as 3 repetitions of a 2-split of 1 unit results in a $\frac{1}{8}u$, and 2 repetitions of a 3-split of 1 unit results in $\frac{1}{9}u$. Students also compose 2- and 3-splits to generate units such as $\frac{1}{6}$ *unit* and $\frac{1}{12}$ *unit*. Split units are compared to whole units and to other split units multiplicatively: $1u$ is 3 times as long as $\frac{1}{3}$ *unit*. $\frac{1}{3}$ *unit* is 2 times as long as $\frac{1}{6}$ *unit*. $\frac{1}{2}$ *unit* is 4 times as long as $\frac{1}{8}$ *unit*. Students can treat arbitrary points on the scale, including fractional units, as if they were zero. For example, with a ruler, a length of $2\frac{1}{2}$ *inches* can be measured either as the distance from 0 *in* to $2\frac{1}{2}$ *in* or as the distance from 8 *in* to $10\frac{1}{2}$ *in*. Eventually, arbitrary zero points are extended to fractional starting points and ending points, as in traveling from $2\frac{1}{2}$ *in* to $7\frac{3}{4}$ *in* represents a distance of $5\frac{1}{4}$ *inches*. These understandings mean that a student can now interpret the meanings of the symbols on a conventional ruler or yardstick.

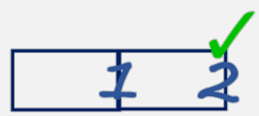


GENERALIZAING RELATIONS

At **ToML-6**, students generalize relations developed for particular units and splits of units. For example, given $\frac{1}{n}$ *unit*, students anticipate that *n* copies generates 1 *unit*. Using knowledge of relations among units, students predict the effects of changes in the unit on measure or scale. For example, if *x-units* are $\frac{1}{2}$ times as long as *y-units*, then a measure of 14 *y-units* will have a measure of 28 *x-units*. Perhaps most significantly, students develop more general understandings of reversible multiplicative relations between units. For example, if unit-*m* is $\frac{a}{b}$ times as long unit-*n*, then the unit-*n* length is $\frac{b}{a}$ times as long as the unit *m* length. So, for a measure of 10*m*, the measure in unit *n* is 15. And for 15*n*, the measure in unit *m* is 10*m*. Students flexibly employ ideas about length measurement to invent units as needed or to decide upon reasonable surrogates, such as time as a stand-in for distance, given knowledge of a particular rate.

Level		Performances		Examples
6	**Generalizing Relations Among Units and Measures**	**6E**	Invent and justify a measure of length.	"We can use the time it takes as a measure of distance, because we can assume a constant rate." "This bushiness index (extent of branching) tells me how much the elodea plant (its total length) grew in the water."
		6D	Anticipate inverse relation between multiplicative comparison of unit lengths.	"$\frac{1}{6}u$ is 2 times as long as $\frac{1}{12}u$. So, $\frac{1}{12}u$ must be $\frac{1}{2}$ times as long as $\frac{1}{6}u$."
		6C	Derive relations among units, given expression of the same attribute in different scales of measure.	"If the measure of the height of the plant is about 10 *cm* or about 4 *inches*, then an inch is about $2\frac{1}{2}$ *cm*."
		6B	Use relations among units to quantify results of changes in unit.	"The measure of the height of the plant is 14 *cm*. If a *cm* is 10 times as long as a *mm*, then the measure is 140 *mm*." "If I change the unit so that it is half as long as the original unit, the measure doubles."
		6A	Anticipate that n iterations of $\frac{1}{n}$ *unit* generate a unit length.	"This length is $\frac{1}{5}$ *Goade*, so it takes 5 of them to make 1 *Goade* unit." (Some students at this level may need to literally iterate to establish this before being able to anticipate it.)
5	**Partitioning and Symbolizing Units Involving 3-splits and Compositions of 2- and 3-Splits**	**5G**	Symbolize multiplicative comparisons involving splits of 3 and 2 with words and arithmetic operations.	"1 *Goade* is 3 times as long as $\frac{1}{3}$ *Goade*." "$\frac{1}{2}$ *inch* is 2 times as long as $\frac{3}{12}$ *inch*." "$\frac{4}{3}$ *Fulk* is 2 times as long as $\frac{2}{3}$ *Fulk*." "$\frac{2}{3}$ *Fulk* is $\frac{1}{2}$ times as long as $\frac{4}{3}$ Fulk."
		5F	Account for change of origin when measurement does not start at zero (fractions and whole numbers).	If I start at 3 and go to $7\frac{1}{4}$, the measure is $4\frac{1}{4}$." "If I travel from $2\frac{1}{2}$ *cm* to $8\frac{3}{4}$ *cm*, it's $6\frac{1}{4}$ *cm*."
		5E	Interpret markings on a standard foot ruler.	Identify inches as $\frac{1}{12}$ *ft*. Understand how different vertical lengths convey 2-splits of inch, such as $\frac{1}{2}$ *in*, $\frac{1}{4}$ *in*, $\frac{1}{8}$ *in*.
		5D	Compose splits of 2 and 3 to generate $\frac{1}{6}$ or $\frac{1}{12}$.	"$\frac{1}{2}$ of $\frac{1}{3}$ *ft* is $\frac{1}{6}$ *ft*." "$\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{3}$ *ft* is $\frac{1}{12}$ *ft*."
		5C	Anticipate outcomes of repetitions of 3-split.	"If you make thirds, and then split it again into thirds, you get 9 parts."

		5B	Symbolize relation between origin and 3 split-partitioned units on scale.	
		5A	Generate a 3-split of a unit and label it as $\frac{1}{3}u$.	"I split it into 3 parts that are exactly the same. Drags finger, saying, "From here (0) to here (first crease of the split unit) is $\frac{1}{3}$ *Goade*. And from 0 to here (second crease) is $\frac{2}{3}$ *Goade*."
4	**Partitioning and Symbolizing Partitioned Units (2- splits)**	**4F**	Account for change of origin when measurement does not start at zero (whole numbers).	"I can start to measure from the 3 on the inches ruler, and take off 3 inches from the result." (Note: Starting at 2 or 3 is generally more difficult than starting at 1. Starting at a non-whole number is more difficult still).
		4E	Symbolize multiplicative comparisons involving splits of 2 with words and arithmetic operations	"$\frac{3}{2}u$ is 3 times as long as $\frac{1}{2}u$" $\frac{1}{2}u$ is 2 times as long as $\frac{1}{4}u$." $1u$ is 4 times as long as $\frac{1}{4}u$." $4 \times \frac{1}{4}u = 1u$ ("4 of $\frac{1}{4}$ u = $1u$") $\frac{1}{2} \times 1u = \frac{1}{2}u$ ($\frac{1}{2}$ of $1u = \frac{1}{2}u$) $\frac{1}{2} \times \frac{1}{2}u = \frac{1}{4}$u ($\frac{1}{2}$ of $\frac{1}{2}u = \frac{1}{4}u$) $2 \times \frac{1}{4}u = \frac{2}{4}u$ (2 of $\frac{1}{4}u = \frac{2}{4}u$)
		4D	Anticipate outcomes of more than 2 repetitions of a 2-split of a unit.	"If I fold the unit to make $\frac{1}{2}$ of the unit, and then fold it in half again, and then again—3 times—the unit will be split into 8 (equal) parts." (then folds unit to show this is true)
		4C	Coordinate whole and part units	"It was 15 inches or $1\frac{1}{4}$ feet." *Note:* These are 2-level units, or units-of-units.  "It's 6 quarters long, or we can say $1\frac{1}{2}$ or $1\frac{2}{4}$ units long."
		4B	Symbolize relation between origin and partitioned units on scale.	"You don't write $\frac{1}{4}$ in the middle. . .  …put it at the end of the part of the unit, so you can see how far you have traveled." 

		4A	Partition and compose equipartitions by factors of 2 (2-split), and use to measure a length.	“It takes 2 and a half units to measure this notebook.” “If you split the unit by 2 and then by 2 again, you get 4 equal parts.”
3	**Iterating Units and Symbolizing Length Measure as Distance Traveled**	**3F**	Iterate composite unit, reflecting 2 levels of unit structure.	“I lined up the zero on my ruler with the start of the path and traveled to the 3 *ft*. mark at the end of the ruler. Then I kept my place and moved the ruler to line up my place with zero and then the end of the path was at 2 *ft*. So the total distance that I traveled was 5 *ft*.” (Note: simultaneous marking of 3*u* and 0)
		3E	Explain how magnitude of measure is the ratio of the number of units accumulated to the unit length.	“ This is 5 *Goades* long. It is 5 times as long as 1 *Goade*.”
		3D	Use and justify standard (including conventional) unit.	“If we all agree to use Justin’s foot as a unit, then we can compare our measurements of the lengths of different desks.”
		3C	Symbolize measure as whole number unit indicating distance traveled.	“You don't write a 1 in the middle of the unit like this one…  …because the unit starts at zero and ends at 1 — that's how far you have traveled! When you put it in the middle, you don’t see where the unit ends!" 
		3B	Symbolize the starting point of measure as zero (0).	When you haven’t traveled yet you start at the starting point, called zero (the heel of the foot), and the toe of your right foot marks 1 foot,. And the next step with your left foot is 2 feet (see the heel of your left foot touches the toe of your right foot).
		3A	Re-use (iterate) a unit to measure.	I just had one unit so I marked its end and then used it again, marked its end again, and kept doing that. It’s 8 paper clips long” Note: Iteration includes both the concepts of translation and accumulating count, and the procedural competence involved in keeping track of the translated unit.
		2F	Qualitatively predict the inverse relation between size of unit and measure.	“If we use small steps, the measure is larger than if we use large steps.

2	**Explaining Properties of Units and Their Role in Accumulation**	**2E**	Consider suitability of unit and explain why.	"That (distance) is very long, so using my clipboard (as a unit) works better than my pencil (as a unit.)"
		2D	Count with reservoir of identical units to tile a length and represent measure by the total. If units are not identical, distinguish among them.	Tiles 8 red unit lengths, counts all, reports measure as "8 reds." Tiles 2 blue, 4 red units. Reports measure as "2 blues, 4 reds."
		2C	Use identical units and explain why.	"It is better to measure with all the same units because then you can just count the number of units." "If you use different units, then you have to tell which ones—like, this line is 2 red and 3 green units long."
		2B	Tile and explain why (the explanation is required).	"The units should touch so there is no gap. Gaps mean that some of the space is not measured." The units should not overlap. If they do, you measure some of the space twice.
		2A	Associate measure with count.	This book is 4 (Student reads number off a ruler.)" "The pencil is 5 paper-clips long (the paper clips may not be identical lengths).
1	**Directly Comparing**	**1F**	Develop and use local (classroom) conventions to distinguish or order two or more objects by a single attribute.	"We decided that for height the streamer started at the floor and went straight up until you could see it was level with the top of the pumpkin (not the stem). Then we found that pumpkin A was taller than C."
		1E	Distinguish (e.g., equal, not equal) or order (e.g., greater, lesser) magnitudes of an attribute by direct comparison of representations.	"Pumpkin A is taller than pumpkin C" (Student aligns paper strips that stand in for height and notices that A's strip is longer than C's strip).
		1D	Distinguish (e.g., equal, not equal) or order (e.g., greater, lesser) magnitudes of an attribute by direct physical comparison.	"This book is taller than that one (Student aligns the books and compares)." (This relation of equality tends to be mastered first.) "Johnny is tallest, and Sally is in the middle, and Jennifer is the shortest." (Ordinal relations tend to be more difficult for very young children).
		1C	Define the attribute being measured.	"Fat means how far it is around the caterpillar" (analogy to circumference of wrist). "Big means the one that weighs the most."
		1B	Identify measurable attributes (qualities).	"We could find out how long the caterpillar is or how fat it is."

		1A	Pose a question or make statements about a potentially measurable object of interest.	▪ "How big is the pumpkin? ▪ "Which rocket flies the best?" ▪ "Which pumpkin is tallest?"